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Thus, in any case,

$$(a + b)(a - b) \equiv 0 \text{ (mod. 8)},$$

a and b each belong to one of the types $3k$, $3k + 1$, $3k + 2$.

If either be $3k$, then

$$ab \equiv 0, \text{ mod. } 3.$$

If both be of type $3k + 1$, or $3k + 2$, then

$$a - b \equiv 0, \text{ mod. } 3.$$

If one be of type $3k + 1$ and the other $3k + 2$, then

$$a + b \equiv 0, \text{ mod. } 3.$$

Thus in any case

$$(a + b)ab(a - b) \equiv 0 \text{ (mod. 24)}.$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

NEW QUESTIONS.

29. While studying the problem of two equal rough bodies, connected by an inelastic wire, resting on an inclined plane, Professor Clifford N. Mills of South Dakota State College met the following interesting expression. If $1/a$ and $1/a + 1$ are the coefficients of friction, the tension of the wire when the bodies are about to descend becomes a multiple of $1/a[1/a + 1/a + 1]$. This, when simplified, becomes $(2a + 1)/2a(a + 1)$. If $2a + 1$ and $2a(a + 1)$ represent the base and altitude of a right triangle the hypothenuse is $2a^2 + 2a + 1$. Therefore this gives a series of numbers which satisfy the relation $x^2 + y^2 = z^2$, if a is given any value whatsoever. Professor Mills desires to know if this will give all the integers which satisfy the condition that the sum of the squares of two integers equals the square of an integer.

REPLIES.

23. What should be done with the theory of limits in elementary geometry? Should the recommendation of the National Committee of Fifteen be universally adopted? If not, what better disposition of the subject can be made?

REPLY BY R. M. MATTHEWS, Riverside, California.

In elementary euclidean plane geometry the theory of limits is used at four places. The last of these is for the evaluation of a particular sequence which defines a definite limit π and so is different from the other three which are the theorems fundamental to proportion and the comparison of areas. The first is, "In a circle, central angles are proportional to their intercepted arcs." It depends upon the theorem that equal central angles intercept equal arcs, a relation which is established by congruence. The second is, "A set of parallel lines intercept proportional segments on all transversals." It depends upon the theorem that parallels which intercept equals on any transversal intercept equals on every transversal, a relation also established by congruence. The third is, "Rectangles are proportional to the products of their bases and altitudes" and its proof is essentially an extension of the theory of measurement implied in the second.

These three propositions can be treated in either of two ways. There is the purely geometric theory of proportion which Euclid developed and which we do not teach. It is not essentially dependent on the idea of ratio or on a complete theory of arithmetic. On the other hand, there is the method of basing geometric proportion on the theory of arithmetic proportion. This latter theory depends on the idea of the ratio of two numbers and connection between arithmetic and geometry is made by defining the ratio of two geometric magnitudes as the ratio of their numerical measures. Accordingly, the whole matter becomes a question of measurement and there would be no difficulty if all magnitudes were "commensurable." This distinction of "commensurable" and "incommensurable" is a survival from Euclid's purely geometric treatment and is needless when a complete system of real numbers is applied to the theory of measurement.

In the measurement of linear segments, the standard distinction is that two segments are commensurable when there exists a segment which can be applied to each an *integral* number of times; otherwise they are incommensurable. Thus, the primary concept is *the number of times one segment is contained in another.* In this primary concept we retain the impression of the simple operation of applying the one segment to the other a step at a time and so argue that "half a time" is absurd. Nevertheless, we practically abandon this primary impression and enlarge our idea of "number of times" when we attach a definite meaning to the statement that segment AB is contained $7\frac{56}{81}$ (for example) times in segment XY. Theoretically we conceive of AB as divided into 81 equal parts and that one of these parts is contained $(7 \times 81 + 56)$ times in XY. The notion of "times contained" has been extended so as to render available all the advantages of the rational number system.

The side and diagonal of a square are declared incommensurable because no segment exists which is contained in both a rational number of times. We need not be confined to rational numbers, however, and the idea of "number of times contained" may be enlarged so as to comprehend every case.

Arithmetically, $\sqrt{2}$ may be defined as the class of all convergent infinite sequences whose limits are the same as the limit of the infinite sequence of rational numbers 1, 1.4, 1.41, 1.414, ...

Geometrically, the side of a square may be applied, in the primary sense, to the diagonal once and a point A_1 determined. A tenth part of the side may be applied 14 times and a point A_2 determined; a hundredth part may be applied 141 times and a point A_3 determined; and so on. This process defines on the diagonal an infinite sequence of points which converges to the end point of the diagonal as a limit. If a different subdivision of the side be used, a different sequence of points would result which, nevertheless, would be equivalent to the first in that it too would converge to the end point of the diagonal.

Thus, on the one hand, there is a class of equivalent convergent sequences of rational numbers, and, on the other hand, a corresponding class of equivalent convergent sequences of points. Dedekind's axiom is to the effect that to the irrational number defined by the class of numerical sequences corresponds one and only one point, namely, that which is the limit point of the corresponding sequences of points. This, it seems to me, is the adequate enlargement of our idea of "number of times contained" so that we can speak as definitely and rigorously of one segment being contained $\sqrt{2}$ times in another segment as we do when we say that one is contained 1.4 times in another. In brief, *for the theory of measurement in elementary geometry the limit process has been applied once for all in the correlation of real numbers with the points on a line.*

The present difficulties are encountered because we are allowing ourselves to be confined to the number system that Euclid knew—and which was adequate for his method of treating proportion—while trying to treat proportion by arithmetic, which he did not do.

According to the foregoing argument, the results of modern research justify us in regarding every two segments as commensurable and, consequently, all theorems on proportional lines, angles and areas will be proved as they now are in the conventional "commensurable case."

It may be asked, however, what must be taught to high school students for the Dedekind axiom and how a revised proof would run. The usual distinction between commensurable and incommensurable would be replaced by the explicit assumption: *For every two like magnitudes there exists a number that denotes the number of times the one is contained in the other.*

Not only is the equivalent of this axiom at the foundation of analysis but it is the very assumption that seems reasonable to the student. Moreover, we do use it. Let us, therefore, make it explicit.

In secondary school work irrationals are introduced in arithmetic and some use is made of them in the first course in algebra, Since we proceed as if our students had the whole real number system, is it unreasonable to apply it to geometry? Possibly our methods of teaching irrational numbers may need revision, but that is a question for arithmetic and algebra rather than for geometry.

With the foundations for a simple and complete theory of measurement laid in the needed assumption, the proofs for the three important theorems would be simplified. In fact, they would be virtually the same as in the ordinary "commensurable case" except that the numbers used would represent any real numbers instead of merely integers.

The case of limits in the measurement of the circle is different, since in that case the difficulty is not with the limit (whose existence is postulated) but with the setting up of a numerical sequence which will evaluate that limit approximately.

COMMUNICATION.

To the Editor of the Monthly: It seems that a criticism which I made in reviewing Dowling and Turneaure's *Analytic Geometry* in the March number of the Monthly was not very clear, since Professor Dowling states in the May number that the proof which I said was lacking is given on page 58, whereas I